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APPROXIMATE SOLUTION OF NONLINEAR PROBLEMS OF OPTIMAL CONTROL OF OSCILLATORY PROCESSES USING THE METHOD OF CANONICAL SEPARATION OF MOTIONS

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L. D. AKULENKO
(Moscow)

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An asymptotic method of solving certain problems of optimal control of motion of the standard type systems with rotating phase is developed. It is assumed that the controls enter only the small perturbing terms, and that the fixed time interval over which the process is being considered is long enough to ensure that the slow variables change essentially. Assuming also that the system and the controls satisfy the necessary requirements of smoothness, the method of canonical averaging [1] is used to construct a scheme for deriving a simplified boundary value problem of the maximum principle. The structure of the set of solutions of the boundary value problem is investigated and a scheme for choosing the optimal solution with the given degree of accuracy in the small parameter is worked out. The validity of the approximate method of solving the boundary value problem is proved. The method suggested in [2] for constructing a solution in the first approximation for similar problems of optimal control is developed.

1. Formulation of the problem of optimal control for the standard systems. We consider a controlled system of standard type with a rotating phase [2]

$$a^{\cdot} = \varepsilon f(a, \psi, u, \varepsilon), \quad a(t_0) = a_0 \tag{1.1}$$
$$\psi^{\cdot} = \omega(a) + \varepsilon F(a, \psi, u, \varepsilon), \quad \psi(t_0) = \psi_0$$

Here $a = (a_1, \ldots, a_n)$ is the slow vector, ψ is the scalar rotating phase, $u \in U$ is the control vector of dimension m, U is the fixed convex set, $|\varepsilon| \leqslant \varepsilon_0$ is a small numerical parameter; t_0, a_0 and ψ_0 are the initial parameters. The functions f and F are assumed to be 2π-periodic in ψ and differentiable a sufficient number of times with respect to all arguments. The control performance criterion is taken in the form of a smooth function of the value of the variable a at the fixed instant of time $t = T$, where $T \sim \varepsilon^{-1}$ [2]. Since the dimension n of the vector a is arbitrary, we can assume that the minimizing functional J has the form

$$J = a_1(T) \to \min\ u \in U \tag{1.2}$$

We require to construct a solution of the optimal control problem (1.1), (1.2) to an arbitrary, predetermined accuracy in the powers of the small parameter. The solution is based on the necessary conditions of the maximum principle [3]. We construct the admissible solutions of the two-point problem to within the prescribed accuracy with respect to the slow variables and the functional, and from amongst these we choose the optimal solution [2].

The boundary value problem consists of solving simultaneously the equations (1.1) and a system for the conjugate variables p and q with the following boundary conditions at $t = T$

$$p^{\cdot} = -\partial H / \partial a = -\omega' q - \varepsilon [\partial (pf) / \partial a + q \partial F / \partial a] \tag{1.3}$$

$$q^{\cdot} = -\partial H / \partial \psi = -\varepsilon [\partial (pf) / \partial \psi + q \partial F / \partial \psi]$$

$$p(T) \equiv p_T = (-1, 0, \ldots, 0), \quad q(T) = 0 \tag{1.4}$$

Here H is the Hamilton function of the problem [3]

$$H = \varepsilon (pf) + q (\omega + \varepsilon F), \quad (pf) = p_1 f_1 + \ldots + p_n f_n \tag{1.5}$$

The control u^* is chosen from the condition of maximum of the function H with respect to u, with the remaining arguments fixed

$$u^* = V(a, \psi, p, q, \varepsilon) \tag{1.6}$$

We assume that the sufficiently smooth function V can be determined uniquely and is 2π-periodic in ψ.

The system of equations (1.1), (1.3) is not standard in the sense of [2], as $p^{\cdot}$ does not vanish identically when $\varepsilon = 0$. However, we can use the approach employed in [2] to reduce this system to the standard form of $2n + 1$ equations in a, ψ and p, since the variable $q \sim \varepsilon$ can be uniquely determined from the condition of constancy in the function H

$$H|_V = \varepsilon h \equiv -\varepsilon A_1 (a(T), \psi(T), p_T, 0, \varepsilon), \qquad h \sim 1 \tag{1.7}$$

$$q = \varepsilon Q(h, a, \psi, p, \varepsilon) \equiv \varepsilon \omega^{-1} [h - (pA_0)] \{1 - \varepsilon \omega^{-1} [(p \partial A / \partial q)_0 + \Psi_0]\} + \varepsilon^3 \ldots$$

Here A and Ψ denote the functions f and F containing the expressions for u given by (1.6) and the subscript $_0$ indicates that the corresponding value of ε is zero.

Thus the solution of the initial boundary value problem is reduced to solving the standard system with a rotating phase for a, ψ and p, after which the parameter h and the variable q can be found from the relations (1.7).

The proof of the validity of the method of averaging the approximate solution of the standard system with a rotating phase is given in [1] for the initial conditions stated above and $h \sim 1$. Sects. 2 and 3 of that paper deal with the problems of constructing the approximate solution of the boundary value problem, and determining the parameter h with the required accuracy. The controlled systems with a small parameter and the systems with small controls acting over short and asymptotically long intervals of time, were investigated in [4 – 11].

2. Constructing an averaged canonical system of equations.

The order of the system can be further reduced by dividing by ψ, and this converts it to the standard form [1]. When the value of h is fixed, the resulting nonautonomous system of $2n$ equations is canonical, and its Hamilton function is $-\varepsilon Q(h, a, \psi, p, \varepsilon)$ [12]

$$da / d\psi = \varepsilon A (\omega + \varepsilon \Psi)^{-1} = -\varepsilon \, \partial Q / \partial p \tag{2.1}$$

$$dp / d\psi = -\varepsilon \{\omega' Q + [\partial (pA) / \partial a + \varepsilon Q \partial \Psi / \partial a]\} (\omega + \varepsilon \Psi)^{-1} = \varepsilon \partial Q / \partial a$$

Here the rotating phase is regarded as an independent variable, while the initial and boundary conditions for a and p as functions of ψ have the form (1.1), (1.4). The value of ψ_0 is known, and ψ_T can be found from the relation

$$t - t_0 = \int_{\psi_0}^{\psi} \frac{d\psi'}{\omega + \varepsilon A}, \quad \psi_T = \psi(T), \quad T - t_0 = L\varepsilon^{-1}, \quad L \sim 1 \tag{2.2}$$

We further assume that the solution of the boundary value problem for the system (2.1), with $h \sim 1$ and $\psi_T \sim \varepsilon^{-1}$, given, belonging to the admissible domain for all $0 < |\varepsilon| \leqslant \varepsilon_0$ exists, and is unique. Let us apply the methods of canonical averaging over the variable ψ [1, 9, 13] to the system (2.1). We perform the change of the initial variables a and p to the new (averaged) variables ξ and η with any prescribed degree of accuracy in ε

$$\xi = a + \varepsilon\, \partial\sigma / \partial\eta, \quad p = \eta + \varepsilon\, \partial\sigma / \partial a \tag{2.3}$$

in such a manner, that the equations remain canonical, and the new Hamiltonian εR does not contain the independent variable ψ

$$d\xi / d\psi = \varepsilon\partial R / \partial\eta, \quad d\eta / d\psi = -\varepsilon\partial R / \partial\xi, \quad R = R(h, \xi, \eta, \varepsilon) \tag{2.4}$$

The generating function periodic in ψ which is nearly equal to the identity $a\eta + \varepsilon\sigma(h, a, \psi, \eta, \varepsilon)$ and the averaged Hamiltonian, can be found with any prescribed degree of accuracy in ε, determined by the smoothness of the system (2.1), in the form

$$\sigma = \sigma_0 + \varepsilon\sigma_1 + \ldots + \varepsilon^k\sigma_k + \ldots, \quad R = R_0 + \varepsilon R_1 + \ldots + \varepsilon^k R_k + \ldots \tag{2.5}$$

from the following partial differential equations [9 — 13]:

$$\partial\sigma / \partial\psi - Q(h, a, \psi, \eta + \varepsilon\partial\sigma / \partial a, \varepsilon) = R(h, a + \varepsilon\partial\sigma / \partial\eta, \eta, \varepsilon) \tag{2.6}$$

Substituting the expressions (2.5) into (2.6) and equating the coefficients of like powers of ε yields, consecutively, the unknown functions σ_i and R_i $(i \geqslant 0)$

$$R_i(h, \xi, \eta) = -\langle Q_i\rangle(h, \xi, \eta) \tag{2.7}$$

$$\langle Q_i\rangle(h, a, \eta) \equiv -\frac{1}{2\pi}\int_0^{2\pi} Q_i(h, a, \psi, \eta)\, d\psi$$

$$\sigma_i(h, a, \psi, \eta) = \int_{\psi_0}^{\psi} [Q_i(h, a, \psi', \eta) - \langle Q_i\rangle(h, a, \eta)]\, d\psi'$$

The square brackets in (2.7) indicate the averaging over ψ The functions Q_i are determined, at each step, in terms of known $Q_0, R_0, \ldots, Q_{i-1}, R_{i-1}$. For example, we have

$$Q_0 = Q(h, a, \psi, \eta, 0), \quad Q_1 = \left(\frac{\partial Q}{\partial\varepsilon}\right) + \left(\frac{\partial Q}{\partial p}\right)\frac{\partial\sigma_0}{\partial a} + \left(\frac{\partial R_0}{\partial\xi}\right)\frac{\partial\sigma_0}{\partial\eta} \tag{2.8}$$

$$Q_2 = \frac{1}{2}\left(\frac{\partial^2 Q}{\partial\varepsilon^2}\right) + \left(\frac{\partial^2 Q}{\partial\varepsilon\,\partial p}\right)\frac{\partial\sigma_0}{\partial a} + \frac{1}{2}\left(\frac{\partial^2 Q}{\partial p^2}\right)\left(\frac{\partial\sigma_0}{\partial a}\right)^2 + \left(\frac{\partial Q}{\partial p}\right)\frac{\partial\sigma_1}{\partial a} + \left(\frac{\partial R_1}{\partial\varepsilon}\right) + \left(\frac{\partial R_1}{\partial\xi}\right)\frac{\partial\sigma_0}{\partial\eta} + \left(\frac{\partial R_0}{\partial\xi}\right)\frac{\partial\sigma_1}{\partial\eta} + \frac{1}{2}\left(\frac{\partial^2 R_0}{\partial\xi^2}\right)\left(\frac{\partial\sigma_0}{\partial\eta}\right)^2$$

where the expressions contained in the parentheses of the type $(\partial Q / \partial\varepsilon)$, $(\partial R_0 / \partial\xi)$ indicate that the functions in question are calculated at $\varepsilon = 0$.

The system (2.4) is appreciably simpler to integrate, since it is canonical and the function R is independent of ψ. Consequently $R(h, \xi, \eta, \varepsilon) = \text{const}$ and, using this integral to eliminate any of the variables, we obtain an autonomous system the order of which can be again reduced by unity. In the concrete cases the above transformations enable us to simplify the analytic construction and the investigation of the solution of the boundary value problem. The system (2.4) is suitable for numerical computations, as we can introduce a slow independent variable $\theta = \varepsilon\psi$ and construct the solution on a relatively short interval: $\theta \in [\theta_0, \theta_T]$ where $\theta_0 = \varepsilon\psi_0 \sim \varepsilon$, and $\theta_T = \varepsilon\psi_T \sim 1$.

3. Approximate solution of the problem of optimal control. Using the formulas (2.7) and (2.8), we write the averaged system of the ($k+1$)-th approximation

$$\frac{d\xi_{(k+1)}}{d\theta} = \sum_{i=0}^{k} \varepsilon^i \frac{\partial R_i}{\partial \eta}, \qquad \frac{d\eta_{(k+1)}}{d\theta} = -\sum_{i=0}^{k} \varepsilon^i \frac{\partial R_i}{\partial \xi} \tag{3.1}$$

in which the ($k+1$)-th and higher order terms in ε are disregarded. It follows that, generally speaking [1], $|\xi - \xi_{(k+1)}| \sim \varepsilon^{k+1}$, $|\eta - \eta_{(k+1)}| \sim \varepsilon^{k+1}$ for $\theta \sim 1$.

1°. A general solution of the system (3.1) can be constructed with an error of the order of ε^{k+1}, using the solution of the first approximation system, i.e. with $k = 0$. A higher approximate solution of the initial boundary value problem is found more suitable if we construct a general solution of the averaged two-point problem (3.1) satisfying the conditions $\xi_{(k+1)}(\theta_0) = c_\xi$ and $\eta_{(k+1)}(\theta_T) = c_\eta$, where c_ξ and c_η are certain arbitrary constants belonging to the ε-neighborhood of the points a_0 and p_T, respectively. We further assume that such a solution of the generating system, i.e. of the first approximation system, is known:

$$\xi_{(1)} = \xi_{(1)}(\theta, \theta_T, h, c_\xi, c_\eta), \quad \eta_{(1)} = \eta_{(1)}(\theta, \theta_T, h, c_\xi, c_\eta) \tag{3.2}$$

The dependence of the known parameter θ_0 is not substantial, hence it is not shown here. The solution sought has the following form in the interval $\theta \sim 1$ [1]:

$$\xi_{(k+1)} = \xi_{(1)} + \sum_{i=1}^{k} \varepsilon^i \delta\xi_{(i)}, \quad \eta_{(k+1)} = \eta_{(1)} + \sum_{i=1}^{k} \varepsilon^i \delta\eta_{(i)} \tag{3.3}$$

The unknown functions $\delta\xi_{(i)}$ and $\delta\eta_{(i)}$ must satisfy the conditions $\delta\xi_{(i)}(\theta_0) = \delta\eta_{(i)}(\theta_T) = 0$, and are determined, successively, from the equations

$$\begin{aligned} \frac{d\delta\xi_{(i)}}{d\theta} &= \left(\frac{\partial^2 R_0}{\partial\eta\,\partial\xi}\right)\delta\xi_{(i)} + \left(\frac{\partial^2 R_0}{\partial\eta^2}\right)\delta\eta_{(i)} + v_{(i)}(\theta, \theta_T, h, c_\xi, c_\eta) \\ \frac{d\delta\eta_{(i)}}{d\theta} &= -\left(\frac{\partial^2 R_0}{\partial\xi^2}\right)\delta\xi_{(i)} - \left(\frac{\partial^2 R_0}{\partial\xi\,\partial\eta}\right)\delta\eta_{(i)} + w_{(i)}(\theta, \theta_T, h, c_\xi, c_\eta) \end{aligned} \tag{3.4}$$

Here the expressions of the type $(\partial^2 R_0 / \partial\eta\partial\xi)$ mean that the derivatives are calculated for the generating solution (3.2), and $v_{(i)}$, $w_{(i)}$ are functions, known for each step, e.g. $v_{(1)} = (\partial R_1 / \partial\eta)$, $w_{(1)} = -(\partial R_1 / \partial\xi)$. The solution of the linear inhomogeneous system (3.4) can be constructed using the method of varying the integration constants, and is based on the fundamental system of solutions X of the corresponding homogeneous system. Taking into account the initial and the boundary conditions, we arrive at the expressions which yield a unique solution of the boundary value problem

$$\left\| \begin{matrix} \delta\xi_{(i)} \\ \delta\eta_{(i)} \end{matrix} \right\| = X b_{(i)} + X \int_{\theta_0}^{\theta} X^{-1} \left\| \begin{matrix} v_{(i)} \\ w_{(i)} \end{matrix} \right\| d\theta' \equiv X b_{(i)} + \left\| \begin{matrix} \beta_{(i)\xi}(\theta) \\ \beta_{\eta(i)}(\theta) \end{matrix} \right\| \tag{3.5}$$

$$X = \left\|\begin{matrix}\partial\xi_{(1)}/\partial c_\xi & \partial\xi_{(1)}/\partial c_\eta \\ \partial\eta_{(1)}/\partial c_\xi & \partial\eta_{(1)}/\partial c_\eta\end{matrix}\right\| \quad b_{(i)} \equiv \left\|\begin{matrix} b_{\xi(i)} \\ b_{\eta(i)}\end{matrix}\right\| = \left\|\begin{matrix} 0 \\ -\beta_{\eta(i)}(\theta_T)\end{matrix}\right\|$$

$$\partial\xi_{(1)}/\partial c_\xi|_{\theta_0} = I,\ \partial\xi_{(1)}/\partial c_\eta|_{\theta_0} = 0,\ \partial\eta_{(1)}/\partial c_\eta|_{\theta_T} = I,\ \partial\eta_{(1)}/\partial c_\xi|_{\theta_T} = 0$$

Here I denotes a unit $n \times n$-matrix, ξ and η are column vectors, and all functions appearing here also depend on the real parameters h, θ_T, c_ξ and c_η. It must be noted that the functions $\delta\xi_{(i)}$ and $\delta\eta_{(i)}$ depend on θ, and are therefore slow. In constructing the required solution $\xi_{(k+1)}$, $\eta_{(k+1)}$ we can utilize the recurrence procedure of the method of consecutive approximations in the powers of ε.

2°. We use (2.3) to construct the solution of the ($k+1$)-th approximation to the initial system (2.1)

$$a_{(k+1)} = \xi_{(1)} + \sum_{i=1}^{k} \varepsilon^i \delta a_i, \quad p_{(k+1)} = \eta_{(1)} + \sum_{i=1}^{k} \varepsilon^i \delta p_i \tag{3.6}$$

The functions $\delta a_{(i)}$ and $\delta p_{(i)}$ which depend on ψ, θ, θ_T, h, c_ξ and c_η, can be determined successively by substituting (3.6) into (2.3) and equating the coefficients of like powers in ε, e.g.

$$\delta a_{(1)} = \delta\xi_{(1)} - (\partial\sigma_0/\partial\eta), \quad \delta p_{(1)} = \delta\eta_{(1)} + (\partial\sigma_0/\partial\xi) \tag{3.7}$$

As the result, we obtain the following approximate expressions for the functions a and p:

$$\begin{aligned} a_{(k+1)} &= \xi_{(1)} + \varepsilon\Delta a_{(k)}(\psi, \theta, \theta_T, h, c_\xi, c_\eta, \varepsilon), \ a_{(k+1)}(\theta_0) = \\ &\quad c_\xi + \varepsilon\Delta a_k(\theta_0) \\ p_{(k+1)} &= \eta_{(1)} + \varepsilon\Delta p_{(k)}(\psi, \theta, \theta_T, h, c_\xi, c_\eta, \varepsilon), \ (p_{(k+1)}(\theta_T) = \\ &\quad c_\eta + \varepsilon\Delta p_k(\theta_T) \end{aligned} \tag{3.8}$$

which represent, as we can see from (3.6) — (3.8), the sums of the smooth functions and of the small oscillatory terms of the order of ε.

3°. The constants of integration c_ξ and c_η must be chosen in such a manner, that the functions $a_{(k+1)}$ and $p_{(k+1)}$ satisfy the prescribed initial and boundary conditions. Since the right-hand sides of the expressions (3.8) are smooth in c_ξ and c_η, we have

$$c_\xi = a_0 + \sum_{i=1}^{k} \varepsilon^i \delta c_{\xi(i)}, \quad c_\eta = p_T + \sum_{i=1}^{k} \varepsilon^i \delta c_{\eta(i)} \tag{3.9}$$

The unknown coefficients $\delta c_{\xi(i)}$ and $\delta c_{\eta(i)}$ depend on the parameters h and θ_T, and periodically on ψ_0 and ψ_T. They can be found by substituting (3.9) into (3.8), expanding with respect to ε which appears in these expressions explicitly, and equating terms of like powers in ε. In particular, we have

$$\begin{aligned}\delta c_{\xi(1)} &= \partial\sigma_0/\partial\eta\,(h, a_0, \psi_0, \eta_{(1)}(\theta_0)), \ \delta c_{\eta(1)} = \\ &-\partial\sigma_0/\partial\xi\,(h, \xi_{(1)}(\theta_T), \psi_T, p_T)\end{aligned} \tag{3.10}$$

Here the functions $\xi_{(1)}$ and $\eta_{(1)}$ have the form (3.2), with $c_\xi = a_0$ and $c_\eta = p_T$. As the result, we obtain the following expressions for c_ξ and c_η

$$c_\xi = a_0 + \varepsilon\Delta c_{\xi(k)}(h, \psi_T, \theta_T, \varepsilon), \ c_\eta = p_T + \varepsilon\Delta c_{\eta(k)}(h, \psi_T, \theta_T, \varepsilon) \tag{3.11}$$

where $\Delta c_{\xi(k)}$ and $\Delta c_{\eta(k)}$ are periodic in ψ_T. This dependence on the known parameters θ_0 and ψ_0 is not shown. Substitution of (3.11) into the expressions (3.8) for $a_{(k+1)}$ and $p_{(k+1)}$ yields

$$a_{(k+1)}^{*} = \xi_{(1)}^{*}(\theta, \theta_T, h) + \varepsilon\Delta a_{(k)}^{*}(\psi, \theta, \psi_T, \theta_T, h, \varepsilon) \qquad (3.12)$$
$$p_{(k+1)}^{*} = \eta_{(1)}^{*}(\theta, \theta_T, h) + \varepsilon\Delta p_{(k)}^{*}(\psi, \theta, \psi_T, \theta_T, h, \varepsilon)$$

where

$$\xi_{(1)}^{*} \equiv \xi_{(1)}(\theta, \theta_T, h, a_0, p_T), \quad \eta_{(1)}^{*} \equiv \eta_{(1)}(\theta, \theta_T, h, a_0, p_T) \qquad (3.13)$$
$$\Delta a_{(k)}^{*}|_{\theta_0} = \Delta p_{(k)}^{*}|_{\theta_T} = 0$$

The functions $\Delta a_{(k)}^{*}$ and $\Delta p_{(k)}^{*}$ are obtained by discarding terms of the order of ε^{k+1} and higher in (3.8). Approximate computation of the coefficients c_ξ and c_η can be carried out using either the method of consecutive approximations in the powers of ε, or the method of tangents.

Thus we obtain the $(k+1)$-th approximations for the functions a and p, satisfying the given initial and boundary conditions. We estimate the nearness of the solution constructed for the boundary value problem (2.1) with ψ_T and h given, using the difference equations $\Delta a = a - a_{(k+1)}^{*}$ and $\Delta p = p - p_{(k+1)}^{*}$ [1]

$$\frac{d\Delta a}{d\psi} = -\varepsilon\left(\frac{\partial^2 Q}{\partial\eta\,\partial\xi}\right)\Delta a - \varepsilon\left(\frac{\partial^2 Q}{\partial\eta^2}\right)\Delta p + \varepsilon R_a + \varepsilon F_a, \quad \Delta a|_{\psi_0} = 0 \qquad (3.14)$$
$$\frac{d\Delta p}{d\psi} = \varepsilon\left(\frac{\partial^2 Q}{\partial\xi^2}\right)\Delta a + \varepsilon\left(\frac{\partial^2 Q}{\partial\xi\,\partial\eta}\right)\Delta p + \varepsilon R_p + \varepsilon F_p, \quad \Delta p|_{\psi_T} = 0$$

Under the assumption adopted in Sect. 2, the solution (3.14) of the boundary value problem vanishes if $R_a = R_p \equiv 0$. Introducing the corresponding column vectors g, r and φ and the square matrix B, we can write the system (3.14) in the form

$$dg / d\psi = \varepsilon Bg + \varepsilon r + \varepsilon\varphi, \quad r = O(\varepsilon^{k+1}), \quad \psi = O(|\varepsilon g| + g^2) \qquad (3.15)$$

Using the transformation

$$g = g_{(1)} + \varepsilon\,[\textstyle\int (B - \langle B\rangle)\,d\psi']g_{(1)} \qquad (3.16)$$

which is almost an identity and in which the integration is carried out only over the explicit argument ψ, we reduce the system (3.15) to the form

$$dg_{(1)} / d\psi = \varepsilon\langle B\rangle g_{(1)} + \varepsilon r_{(1)} + \varepsilon\varphi_{(1)}, \quad r_{(1)} = O(\varepsilon^{k+1}) \qquad (3.17)$$
$$\varphi_{(1)} = O(|\varepsilon g_{(1)}| + g_{(1)}^2)$$

Since the functions $r_{(1)}$ and $\varphi_{(1)}$ obtained from r and φ, respectively, are unwieldy, we omit them from here.

It can be shown that $g_{(1)} = O(r_{(1)})$ for $\psi \in [\psi_0, \psi_T]$, $\psi_T \sim \varepsilon^{-1}$. In fact, the solution of the boundary value problem for the system (3.17) can be found by solving a set of $2n$ integral equations (see (3.5))

$$g_{(1)} = Xl_{(1)} + X\varepsilon\int_{\psi_0}^{\psi} X^{-1}(r_{(1)} + \varphi_{(1)})\,d\psi' \qquad (3.18)$$

Here the vector $l_{(1)}$ is determined similarly as $b_{(i)}$ in (3.5), and its components represent linear functionals of the vector $(r_{(1)} + \varphi_{(1)})$. We construct the solution of the system (3.18), using the method of consecutive approximations in the powers of ε. When $|\varepsilon|$ is sufficiently small, the nonlinear integral operator of $g_{(1)}$ (3.18) has a unique fixed point, i.e. the consecutive approximations converge uniformly to a unique solution $g_{(1)}^{*} = O(r_{(1)})$ [14]. Then from (3.16) it follows that Δa, $\Delta p \sim \varepsilon^{k+1}$ for given $\psi_T \sim \varepsilon^{-1}$ and h.

4°. The parameter ψ_T and function $\psi(t, h, \varepsilon)$ can be found with an error of order $O(\varepsilon^k)$, using the formulas (2.2). The relation for determining the quantity $\theta(t, h, \varepsilon)$ in the $(k+1)$-th approximation assumes the following form after substituting the expressions (3.12), (3.13) and discarding the terms of higher order of smallness in ε:

$$\int_{\theta_0}^{\theta(k+1)} \left[1 + \varepsilon \Delta\Psi_{(k)}\left(\frac{\theta}{\varepsilon}, \theta, \frac{\theta_{T(k+1)}}{\varepsilon}, \theta_{T(k+1)}, h, \varepsilon\right)\right] \frac{d\theta}{\omega(\xi^*_{(1)})} = \tau \qquad (3.19)$$

The quantities θ and θ_T are obtained with an error of $O(\varepsilon^{k+1})$. When $k \geqslant 1$, we have

$$\Delta\Psi_{(k)} = \sum_{i=0}^{k-1} \varepsilon^i \Psi_i(\psi, \theta, \psi_{T(k)}, \theta_{T(k+1)}, h) \qquad (3.20)$$

As was said before, the dependence of ψ and ψ_T is of periodic character. Substitution of (3.20) into (3.19) carried out at $\tau = L$, yields the following expression for the parameter $\theta_{T(k+1)}$:

$$M_{(k+1)}(\theta_{T(k+1)}, h, \varepsilon) \equiv \int_{\theta_0}^{\theta_{T(k+1)}} (1 + \varepsilon\Delta\Psi_{(k)}) \frac{d\theta}{\omega(\xi^*_{(1)})} = L \qquad (3.21)$$

The solution of the transcendental equation (3.21) is obtained in the form of an asymptotic expansion

$$\theta_{T(k+1)}(h, \varepsilon) = \theta_{T(1)}(h) + \sum_{i=1}^{k} \varepsilon^i \delta\theta_{T(i)} \equiv \theta_{T(1)} + \varepsilon\Delta\theta_{T(k)} \qquad (3.22)$$

In the first approximation, the parameter θ_T is found from Eq. (3.21) with $\varepsilon = 0$: $M_{(1)}(\theta_{T(1)}, h) = 0$. Let the solution $\theta_{T(1)}(h)$ exist and $\partial M_{(1)} / \partial\theta_{T(1)} \neq 0$ for this particular value of the root. Then the increment $\delta\theta_{T(1)}$ is given by

$$\frac{\partial M_{(1)}}{\partial\theta_{T(1)}} \delta\theta_{T(1)} = -\int_{\theta_0}^{\theta_{T(1)}} \langle\delta\Psi_{(1)}\rangle(\theta, \psi_{T(0)} + \delta\theta_{T(1)}, \theta_{T(1)}, h) \times \frac{d\theta}{\omega(\xi^*_{(1)}(\theta, \theta_{T(1)}, h))} \qquad (3.23)$$

The above expression certainly has a solution, since it contains a bounded periodic function $\delta\theta_{T(1)}$ in its right-hand side. If $\delta\theta_{T(1)}$ is a simple root of (3.23), then all coefficients of $\delta\theta_{T(i)}$ are found successively from linear equations of the form

$$\left[\frac{\partial M_{(1)}}{\partial\theta_{T(1)}} + \int_{\theta_0}^{\theta_{T(1)}} \frac{\partial\langle\Psi_{(1)}\rangle}{\partial\delta\theta_{T(1)}} \frac{d\theta}{\omega(\xi^*_{(1)})}\right] \delta\theta_{T(i)} = \vartheta_i(h), \quad i = 2, \ldots, k$$

where $\vartheta_i(h)$ are known functions of h, e.g.

$$\vartheta_2 = -\int_{\theta_0}^{\theta_{T(1)}} \left\{\left[\langle\delta\Psi_2\rangle + \delta\Psi_{(1)} - \langle\delta\Psi_{(1)}\rangle\right]\frac{1}{\omega} - \frac{\omega'\xi^{*''}_{(1)}}{2\omega^2}\delta\theta^2_{T(1)} + \left(\frac{\omega'^2}{\omega^3} - \frac{\omega''}{2\omega^2}\right)\xi^{*'2}_{(1)}\delta\theta^2_{T(1)}\right\} d\theta$$

The approximate solution $\theta_{(k+1)}$ of (3.19) can be constructed in a similar manner. In particular, the first approximation $\theta_{(1)}(\tau, h)$ is obtained uniquely from (3.19) with $\varepsilon = 0$. The remaining coefficients are calculated consecutively, e.g.

$$\delta\theta_{(1)} = \omega(\xi^*_{(1)}) \int_{\theta_0}^{\theta_{(1)}} \left[\omega'(\xi^*_{(1)}) \frac{\partial \xi^*_{(1)}}{\partial \theta_{T(1)}} \delta\theta_{T(1)} + \langle \delta\Psi_{(1)}\rangle (0, \psi_{T(0)} + \delta\theta_{T(1)}, \theta_{T(1)}, h)\right] \frac{d\theta}{\omega^2(\xi^*_{(1)})}$$

Thus, we find that for a given value of the parameter h, the admissible solutions of the boundary value problem (1.1), (1.3), (1.4), (1.6), (1.7) in the ($k+1$)-th approximation are constructed in the form of (3.12) in which the values obtained for $\theta_{k+1}(t, h, \varepsilon)$ and $\theta_{T(k+1)}(h, \varepsilon)$ have been substituted.

5°. The parameter h must satisfy the condition of transversality for the variable q (1.7), and its value must be computed with the accuracy of $\sim \varepsilon^{k+1}$ in order that the degree of accuracy indicated in the above relations is really attained. For the essentially nonlinear system considered here, it was established in [2] that $\partial\psi_T / \partial h \sim \varepsilon^{-1}$, therefore the value of h sought is determined (nonuniquely) from the equation $h = -A_1(a^*_{T(k+1)}, \psi_{T(k)}, p_T, 0, \varepsilon)$, since the error $\sim\varepsilon^k$ in determining ψ_T can be compensated by varying the value of h by the amount $\sim\varepsilon^{k+1}$ which represents the admissible error of the computation. Further, it was shown in [2] that a root of the first approximation equation exists equal to $h_{(1)} = -\langle A_1\rangle (\xi^*_{T(1)}, p_T, 0, 0)$. If this root is simple, then at a sufficiently small $|\varepsilon|$ the initial equation admits a discrete set of roots $\{h\}$. As $\varepsilon \to 0$, these roots fill a certain continuous interval $[h_1, h_2]$ which includes the point $h_{(1)}$ (see Fig. 1).

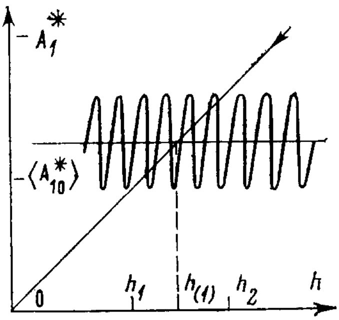

Fig. 1

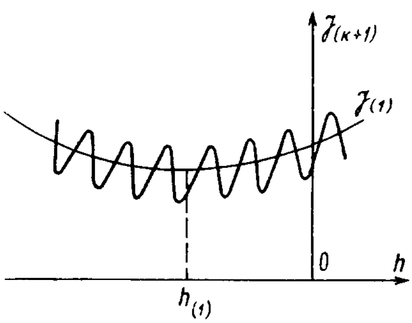

Fig. 2

The ($k+1$)-th order approximate solution of the problem of optimal control (1.1), (1.2) consists now of choosing such a value $h^* \in \{h\}$ in the ε-neighborhood of the point $h_{(1)} \in [h_1, h_2]$, that the functional (1.2) computed with an error of order ε^{k+1} reaches a minimum

$$J_{(k+1)}[h] = a^*_{1T(k+1)} \to \min h \in \{h\} \qquad (3.24)$$

Figure 2 depicts a typical form of the function $J_{(k+1)}[h]$ in the neighborhood of a local minimum. Substituting the expressions obtained in (1.6), we construct the approximate control law

$$u^*_{(k+1)} = V(a, \psi, p^*_{(k+1)}, q^*_{(k+1)}, \varepsilon) \qquad (3.25)$$

All the above arguments are formulated in the following assertion.

Theorem. Let the following conditions hold.

1) The problem of optimal control (1.1), (1.2) has a unique solution belonging to the admissible region for $0 < |\varepsilon| \leqslant \varepsilon_0$.

2) The right-hand sides of the system (2.1) possess continuous partial derivatives with respect to a, p and ε, of up to the ($k+1$)-th order.

3) The boundary value problem (2.1) admits a unique solution, and the boundary value problem of the first approximation has the property of stability described in Sect. 3, 1°.

4) The condition of "rapid oscillation" is fulfilled in the first approximation: $\partial\theta_{T(1)}(h_{(1)}) / \partial h \neq 0$.

Then the functions (3.12) yield a solution of the problem of optimal control (1.1), (1.2) with an error of the order $O(\varepsilon^{k+1})$ in the time interval $T - t_0 \sim L\varepsilon^{-1}$ in terms of the slow variable a and the functional J. The approximate optimal control and the minimum value of the functional have the form (3.25) and (3.24), respectively.

It must be noted that the method developed here can be used to solve other problems of optimal control by slow variables. Such a case occurs when the system (1.1) has to be transferred to a smooth set independent of the rapid variable, at some fixed instant of time $t = T \sim \varepsilon^{-1}$.

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REFERENCES

1. Volosov, V. M. and Morgunov, B. I., Method of Averaging in the Theory of Nonlinear Oscillatory Systems. Izd. MGU, Moscow, 1971.

2. Akulenko, L. D. and Chernous'ko, F. L., Method of averaging in the problems of optimal control. (English translation), Pergamon Press, Zh. vychisl. Mat. mat. Fiz., Vol. 15, № 4, 1975.

3. Pontriagin, L. S., Boltnianskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., The Mathematical Theory of Optimal Processes. (English translation), Pergamon Press, Book № 10176, 1964.

4. Grodzovskii, G. L., Ivanov, Iu. N. and Tokarev, V. V., Mechanics of Low Thrust Space Flights. Moscow, "Nauka", 1966.

5. Moiseev, N. N., Elements of the Theory of Optimal Systems. Moscow, "Nauka", 1975.

6. Krasovskii, N. N., Theory of Motion Control. Moscow, "Nauka", 1968.

7. Chernous'ko, F. L., Some problems of optimal control with a small parameter. PMM Vol. 32, № 1, 1968.

8. Evtushenko, Iu. G., Approximate calculation of optimal control problems. PMM Vol. 34, № 1, 1970.

9. Akulenko, L. D., Investigation of certain optimal systems by the averaging method. PMM Vol. 38, № 3, 1974.

10. Powers, W. F. and Tapley, B. D., Canonical transformation applications to optimal trajectory analysis. AIAA Journal, Vol. 7, № 3, 1969.

11. Jacobson, R. A. and Powers, W. F., Asymptotic solution to the problem of optimal low-thrust energy increase. AIAA Journal, Vol. 10, № 12, 1972.

12. Arnol'd, V. I., Mathematical Methods of Classical Mechanics. Moscow, "Nauka", 1974.

13. Fedorchenko, A. M., Method of canonical averaging in the theory of nonlinear

oscillations. Ukr. matem. Zh. Vol. 9, № 2, 1957.
14. Zabreiko, P. P. et al., Integral Equations. SMB, Moscow, "Nauka", 1968.

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OPTIMAL STABILIZATION OF LINEAR STOCHASTIC SYSTEMS

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G. N. MIL'SHTEIN and L. B. RIASHKO
(Sverdlovsk)
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The problem of optimal stabilization is solved for controlled linear systems with white noise. The optimal solution is obtained by the method of successive approximations each of which represents the optimal solution of the related determinate problem. Necessary and sufficient conditions of stabilizability are given.

1. Formulation and transformation of the problem. Consider a stochastic controlled system of the form

$$\frac{dx}{dt} = \left(A + \sum_{r=1}^{k} \sigma_r \xi_r^{\cdot}\right) x + \left(b + \sum_{r=1}^{m} \varphi_r \eta_r^{\cdot}\right) u \tag{1.1}$$

Here x is an n-dimensional phase coordinate vector, u is the scalar control, A and σ_r are constant $(n \times n)$-matrices, b and φ_r are constant n-vectors, $\xi_r^{\cdot}(t)$ $(r = 1, \ldots, k)$ denote the noise present in the object and $\eta_r^{\cdot}(t)$ $(r = 1, \ldots, m)$ is the noise present in the control channel. In addition, all $\xi_r(t)$ and $\eta_r(t)$ are standard Wiener processes independent within the set.

Let us consider a problem of optimal stabilization [1 — 5] of the system (1.1) with the quality criterion

$$I(u) = M \int_0^\infty [x^* G x + \lambda u^2]\, dt, \quad \lambda > 0 \tag{1.2}$$

where G is a positive definite $(n \times n)$-matrix $(G > 0)$.

If the Bellman function associated with the problem (1.1), (1.2) is sought as a positive definite quadratic form $x^* M x$, the matrix $M > 0$ satisfies the equation

$$A^* M + MA + \sum_{r=1}^{k} \sigma_r^* M \sigma_r - \frac{Mbb^* M}{\lambda + \varphi(M)} = -G \tag{1.3}$$

$$\varphi(M) = \sum_{r=1}^{m} \varphi_r^* M \varphi_r$$

and the optimal control is given by the formula

$$u_0(x) = -b^* M x / (\lambda + \varphi(M)) \tag{1.4}$$

Following [3, 4], we perform the following change of variables in (1.3):

$$D = M / (\lambda + \varphi(M)) \tag{1.5}$$

This yields the system